Available at WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE DIRECT

Journal of Approximation Theory 123 (2003) 89–109

JOURNAL OF Approximation Theory

http://www.elsevier.com/locate/jat

Uniqueness of periodic best L^1-approximations

Manfred Sommer

Mathematisch-Geographische Fakultät, Katholische Universität Eichstätt, D-85071 Eichstätt, Germany

Received 9 July 2002; accepted in revised form 2 April 2003

Communicated by Günther Nürnberger

Abstract

In this paper we give a characterization of the finite-dimensional subspaces of periodic, real-valued and continuous functions which admit uniqueness of best L^1-approximations. Our investigations are based on the well-known Property A which characterizes a finite-dimensional subspace of continuous functions to be a unicity subspace with respect to a class of weighted L^1-norms.

Keywords: Uniqueness of best L^1-approximations; Periodic functions; Property A; Spline functions

1. Introduction

Let C_{b-a} denote the subspace of all continuous, $(b-a)$-periodic functions $f:\mathbb{R}\to\mathbb{R}$ where $a<b$, i.e.,

$$C_{b-a} = \{f\in C(\mathbb{R}): f(x) = f(x+(b-a)),\ x\in\mathbb{R}\}.$$

We are interested in a characterization of the finite-dimensional subspaces U of C_{b-a} such that every $f\in C_{b-a}$ has a unique best approximation from U with respect to a class of weighted L^1-norms. The central role in our investigations plays Property A (Definition 1), introduced by Strauss [7] as a sufficient condition for $L^1(\mu)$-unicity subspaces of real-valued continuous functions defined on $[a,b]$ where $\mu=\lambda$, the Lebesgue measure. In a series of papers written by Kroó, Pinkus, Schmidt, Sommer, Wajnryb (a detailed survey of the results has been given by Pinkus in his excellent monograph [4]), and by Li [2], Property A was applied to give a characterization of $L^1(\mu)$-unicity subspaces of real-valued continuous functions defined on certain

E-mail address: manfred.sommer@ku-eichstaett.de.

0021-9045/03/$ - see front matter
doi:10.1016/S0021-9045(03)00066-2

compact subsets of $\mathbb{R}^d$ $(d \geq 1)$ for a class of 'admissible' measures (Theorem 1). Recently, Babenko et al. (see e.g. [1] for references) also obtained interesting results on uniqueness of best L^1-approximations.

Since every real halfopen interval $[a, b)$ is homeomorphic to the unit sphere S in $\mathbb{R}^2$, the problem of uniqueness of best $L^1(\mu)$-approximations of $f \in C_{b-a}$ from a subspace U of C_{b-a} can be considered as an $L^1(\mu)$-approximation problem in $C(S)$, the space of all real-valued continuous functions on S. In fact, using some general necessary conditions for Property A due to Pinkus and Wajnryb [5] we are able to give a characterization of the finite-dimensional subspaces U of C_{b-a} such that every $f \in C_{b-a}$ has a unique best $L^1(\mu)$-approximation for a class of weighted measures μ (Theorem 5).

Finally, we present some examples for $L^1(\mu)$-unicity subspaces in C_{b-a}, including spaces of trigonometric polynomials, of piecing together Haar systems and of periodic polynomial splines. In particular, we obtain a result of Meinardus and Nürnberger [3] who showed that every function $f \in C_{b-a}$ has a unique L^1-approximation (with respect to λ) from $U = P_m(K_n)$, the subspace of periodic polynomial splines of degree $m \geq 1$ with a set of simple knots K_n.

2. Property A in the nonperiodic case

A central role in best $L^1(\mu)$-approximation problems plays Property A. To define it in a general setting, let $K \subset \mathbb{R}^d$ $(d \geq 1)$ such that

(1) K is a compact set,
(2) $K = \overline{\text{int}\, K}$ (the closure of its interior).

U will always denote an n-dimensional subspace of $C(K)$, the space of all real-valued continuous functions defined on K. We define a set W of measures on K by

$$W = \{\mu : d\mu = w\, d\lambda,\ w \in L^\infty(K),\ \text{ess inf}\ w > 0 \text{ on } K\}$$

(λ means the Lebesgue measure on K). For $\mu \in W$, the $L^1(\mu)$-norm is defined by

$$\|f\|_\mu = \int_K |f|\, d\mu \quad (f \in C(K)).$$

Let $C_1(K, \mu)$ denote the linear space $C(K)$ endowed with norm $\|\cdot\|_\mu$. We say that U is a *unicity space for* $C_1(K, \mu)$, $\mu \in W$, if to each $f \in C(K)$ there exists a unique best approximation from U in the norm $\|\cdot\|_\mu$.

We need some notations as follows. Let for any $g \in C(K)$ and any subset V of $C(K)$,

$$Z(g) = \{x \in K : g(x) = 0\},$$
$$Z(V) = \{x \in K : v(x) = 0 \text{ for all } v \in V\},$$
$$\text{supp}\, V = K \backslash Z(V).$$

Let us now define Property A (cf. [4, p. 98] for its history).

Definition 1. We say that U satisfies *Property A* if to each nonzero $u \in U$ and $u^* \in C(K)$ such that $|u^*| = |u|$ on K there exists a $\tilde{u} \in U \backslash \{0\}$ for which

(1) $\tilde{u} = 0$ a.e. on $Z(u)$ (with respect to λ),
(2) $\tilde{u}u^* \geqslant 0$ on K.

Property A is closely related to the problem of existence of unicity spaces for $C_1(K, \mu)$. In fact, it gives a characterization of such subspaces with respect to every $\mu \in W$.

Theorem 1 (See Pinkus [4, p. 58]). *A finite-dimensional subspace U of $C(K)$ is a unicity space for $C_1(K, \mu)$ for all $\mu \in W$ if and only if U satisfies Property* A.

It should be noted that this result holds for a bigger class of 'admissible' measures which are absolutely continuous with respect to λ.

Various consequences of Property A which, in particular, are very helpful for our periodic problem were obtained. To describe them we need some definitions.

Definition 2. Let $D \subset K$, D (relatively) open. Then $[D]$ will denote the number (possibly infinite but necessarily countable) of open connected components of D.

Definition 3. We say that U *decomposes* if there exist subspaces V and $\tilde{V}$ of U with $\dim V \geqslant 1$, $\dim \tilde{V} \geqslant 1$ such that

(1) $U = V \oplus \tilde{V}$, i.e., $U = V + \tilde{V}$ and $V \cap \tilde{V} = \{0\}$,
(2) $\operatorname{supp} V \cap \operatorname{supp} \tilde{V} = \emptyset$.

To simplify the notations we also define:

Definition 4. For $u \in U$, set

$$U(u) = \{v : v \in U,\ v = 0 \ a.e. \ on \ Z(u)\}.$$

The following consequences of Property A due to Pinkus and Wajnryb are very important to our investigations.

Theorem 2 (See Pinkus [4, Theorems 4.6, 4.12]). *Suppose that U satisfies Property* A. *Then*

(1) $[K \backslash Z(u)] \leqslant \dim U(u)$ *for every* $u \in U$, *and*
(2) U *decomposes, if* $[K \backslash Z(U)] \geqslant 2$.

Remark 1. (1) It is easily seen that if U decomposes by subspaces V and $\tilde{V}$, then U satisfies Property A if and only if both V and $\tilde{V}$ satisfy Property A [4, p. 70].

(2) In particular, Pinkus showed that if $K \subset \mathbb{R}$, the first statement of Theorem 2 is both necessary and sufficient for U to satisfy Property A [4, p. 75].

(3) For the case when $K \subset \mathbb{R}$, Pinkus gave an interesting classification of all finite-dimensional subspaces U of $C(K)$ which satisfy Property A. As a result, such a space has to have a 'spline-like' structure [4, p. 75]. A slightly simplified characterization of such spaces on $K = [a, b]$ was obtained by Li [2].

3. $L^1(\mu)$-approximation by subspaces of periodic functions

Assume now that U and W will denote an n-dimensional subspace of C_{b-a} and the set of weighted measures on $K = [a, b]$ defined in Section 2, respectively.

We say that U is a *periodic unicity space for* $C_1([a, b], \mu)$, $\mu \in W$, if to each $f \in C_{b-a}$ there exists a unique best approximation from U on $[a, b]$ in the norm $\|\cdot\|_\mu$.

Since every function in C_{b-a} is defined on $\mathbb{R}$ and has period $b - a$, our approximation problem can be 'shifted' to any interval $[\alpha, \beta]$ with $\beta - \alpha = b - a$ by extending every measure $\mu \in W$, i.e., $d\mu = w\, d\lambda$, to a 'periodic' measure $\tilde{\mu}$ such that

$$d\tilde{\mu} = \tilde{w}\, d\lambda$$

and

$$\tilde{w}(x) = \begin{cases} w(x) & \text{if } x \in [a, b], \\ w(x + (b - a)) & \text{otherwise.} \end{cases}$$

This implies that if $f \in C_{b-a}$,

$$\min_{u \in U} \int_a^b |f - u|\, d\mu = \min_{u \in U} \int_\alpha^\beta |f - u|\, d\tilde{\mu}.$$

Moreover, to apply some of the statements of the nonperiodic case in Section 2, we consider our periodic approximation problem as a nonperiodic problem on $C(S)$ where S denotes the unit sphere in $\mathbb{R}^2$. In fact, both problems are actually the same, because every halfopen interval $[a, b)$ is homeomorphic to S, for instance by the mapping $\varphi : [a, b) \to S$ defined by

$$\varphi((1 - t)a + tb) = (\cos 2\pi t, \sin 2\pi t), \quad t \in [0, 1).$$

In particular, φ defines a counterclockwise order on S setting $\varphi(c) < \varphi(d)$ if $a \leqslant c < d < b$. Thus, to simplify the following arguments, we identify (if necessary) the function $f \in C_{b-a}$ and the subspace U of C_{b-a} with a function and a subspace of $C(S)$, again denoted by f and U, respectively. It should be noted that for $\mu \in W$ the $L^1(\mu)$-norm of $f \in C_{b-a}$, taken over $[a, b]$ and S, respectively, differs only by a constant factor independently of f.

Although for the compact set $K = S \subset \mathbb{R}^2$ the additional assumption that $K = \overline{\operatorname{int} K}$ does not hold (in the topology of $\mathbb{R}^2$), some of the statements in Section 2 remain valid. In fact, the following statements still hold.

Theorem 3. *A finite-dimensional subspace U of C_{b-a} is a periodic unicity space for $C_1([a,b],\mu)$ for all $\mu \in W$ if and only if U (as a subspace of $C(S)$) satisfies Property* A *on S.*

Proof. Following the lines of the proof of Theorem 1 it turns out that the arguments are also true in the case when $U \subset C(S)$. Thus the statement follows immediately from Theorem 1. □

Remark 2. (1) To make clearer the difference between the statements that U satisfies Property A on $[a,b]$ (which corresponds to the nonperiodic case) and Property A on S (U considered as subspace of $C(S)$), respectively, we give the following definition: We say that the subspace U of C_{b-a} satisfies *Property* $\mathrm{A}_{\mathrm{per}}$ if to each nonzero $u \in U$ and $u^* \in C_{b-a}$ such that $|u^*| = |u|$ on $[a,b]$ there exists a $\tilde{u} \in U \backslash \{0\}$ for which

(1) $\tilde{u} = 0$ a.e. on $Z(u)$,
(2) $\tilde{u}u^* \geqslant 0$ on $[a,b]$.

Thus, U satisfies Property A on S if and only if U satisfies Property $\mathrm{A}_{\mathrm{per}}$.

(2) It is easily seen that if U satisfies Property $\mathrm{A}_{\mathrm{per}}$, then $U(u)$ satisfies Property $\mathrm{A}_{\mathrm{per}}$ for every $u \in U$.

Theorem 4. *Suppose that U satisfies Property* $\mathrm{A}_{\mathrm{per}}$. *Then*

(1) $[S \backslash Z(u)] \leqslant \dim U(u)$ *for every* $u \in U$, *and*
(2) U *decomposes, if* $[S \backslash Z(U)] \geqslant 2$.

Proof. Identify again U with a subspace of $C(S)$. Then $U(u)$ corresponds to a subspace of $C(S)$ for every $u \in U$, and $Z(u)$, $Z(U)$ correspond to subsets of S. Now following the lines of the proof of Theorem 2 it turns out that the same arguments can be applied to the case when $U \subset C(S)$. Thus the statement follows from Theorem 2. □

Remark 3. (1) Of course, Property $\mathrm{A}_{\mathrm{per}}$ is weaker than Property A on $K = [a,b]$. For instance, let $K = [0,1]$ and let $U = \mathrm{span}\{u_1, u_2\} \subset C_{1-0}$ where $u_1(x) = 1$ and $u_2(x) = (x - \frac{1}{4})(x - \frac{3}{4})$, $x \in [0,1]$. Then it follows that $[K \backslash Z(u_2)] = 3$ which, in view of Theorem 2, implies that U does not satisfy Property A on $[0,1]$.

But, considering u_2 as a function on S, it obviously follows that $[S \backslash Z(u_2)] = 2 = \dim U(u_2) = \dim U$. In fact, we can show that U satisfies Property $\mathrm{A}_{\mathrm{per}}$. Suppose that $u = c_1u_1 + c_2u_2 \in U \backslash \{0\}$. Let $u^* \in C_{1-0}$ with $|u^*| = |u|$. Assume first that u has no sign change on $(0,1)$. Then u^* has no sign chance on (0,1) and $\varepsilon uu^* \geqslant 0$ on $[0,1]$ for some $\varepsilon \in \{-1,1\}$. Assume now that u has a sign change $\tilde{x} \in (0,1)$. Then by definition of u_1 and u_2, $Z(u) = \{\tilde{x}, 1-\tilde{x}\}$. This implies that either $\varepsilon u^* \geqslant 0$ or $\varepsilon u^* = u$ on $[0,1]$ for some $\varepsilon \in \{-1,1\}$ (recall that $u^*(0) = u^*(1)$). Then in the first case, $\varepsilon u_1 u^* \geqslant 0$ while in the second case, $\varepsilon uu^* \geqslant 0$ on $[0,1]$.

Thus it follows from Theorems 3 and 1, respectively, that for every $f \in C_{1-0}$ and each $\mu \in W$ there exists a unique best $L^1(\mu)$-approximation from U, and there must exist $\tilde{f} \in C[0,1]$ and $\tilde{\mu} \in W$ such that $\tilde{f}$ fails to have a unique best $L^1(\tilde{\mu})$-approximation from U.

(2) To obtain the same number of connected components of $S \backslash Z(u)$ and $[a,b] \backslash Z(u)$, respectively, we use the periodic properties: Let $u \in U \subset C_{b-a}$ and assume first that $Z(u) = \emptyset$. Then obviously, $[S \backslash Z(u)] = [K \backslash Z(u)] = 1$ where $K = [a,b]$. Assume now that $Z(u) \neq \emptyset$. Let $\tilde{x} \in Z(u)$ and consider u on $\tilde{K} = [\tilde{x}, \tilde{x} + b - a]$. Since $u \in C_{b-a}$, we have $u(\tilde{x} + b - a) = 0$. This implies that

$$[S \backslash Z(u)] = [\tilde{K} \backslash Z(u)].$$

Thus, statement (1) of Theorem 4 is also satisfied replacing S by an interval $\tilde{K}$ which depends on u.

4. Characterization of Property $\mathrm{A}_{\mathrm{per}}$

In the nonperiodic case the inequality

$$[K \backslash Z(u)] \leqslant \dim U(u) \tag{4.1}$$

for every $u \in U$ is both necessary and sufficient for U to satisfy Property A if $K \subset \mathbb{R}$ (see Remark 1). The sufficiency is not true for periodic approximation in general as the following example will show.

Example 1. Let $K = [0,1]$ and assume that $U = \operatorname{span}\{u_1, u_2\} \subset C_{1-0}$ where $u_1(x) = (x - \frac{1}{4})(x - \frac{3}{4})$ and $u_2(x) = x(x - \frac{1}{2})(x-1)$, $x \in [0,1]$. Let $u = c_1 u_1 + c_2 u_2 \in U$. We first show that $[S \backslash Z(u)] \leqslant 2$. This is obviously true if $c_1 = 0$ or $c_2 = 0$. Therefore, assume that $c_i \neq 0$, $i = 1, 2$. Without loss of generality, let $c_1 = 1$ and $c_2 < 0$. This implies that $u(1) = u_1(1) > 0$. Since u coincides on $[0,1]$ with the polynomial

$$p(x) = (x - \tfrac{1}{4})(x - \tfrac{3}{4}) + c_2 x (x - \tfrac{1}{2})(x - 1), \quad x \in \mathbb{R}$$

and $\lim_{x \to \infty} p(x) = -\infty$, it follows that p has a zero in $(1, \infty)$. Thus, u can have at most two zeros in $[0,1]$ (in fact, it has two) and, therefore,

$$[S \backslash Z(u)] \leqslant 2.$$

We now show that U fails to satisfy Property $\mathrm{A}_{\mathrm{per}}$. On the contrary assume that U has this property. Then, since $u^* = |u_2| \in C_{1-0}$, there must exist a $\tilde{u} \in U$ with $u^* \tilde{u} \geqslant 0$, i.e., $\tilde{u} \geqslant 0$ on $[0,1]$. Let $\tilde{u} = c_1 u_1 + c_2 u_2$. Then $c_1 \neq 0$, because u_2 changes the sign on $(0,1)$, and it follows that $\operatorname{sign} \tilde{u}(0) = \operatorname{sign} c_1$ and $\operatorname{sign} \tilde{u}(\frac{1}{2}) = -\operatorname{sign} c_1$, a contradiction.

This shows that statement (4.1) fails to be a sufficient condition for Property $\mathrm{A}_{\mathrm{per}}$ in general.

We now characterize all U in C_{b-a} which satisfy Property $\mathrm{A_{per}}$. On the basis of Theorem 4 we only have to treat the cases $Z(U) = \emptyset$, $Z(U) = \{a, b\}$ and $Z(U) = \{\tilde{x}\}$ for some $\tilde{x} \in (a, b)$, respectively. Since we identify U with a subspace of $C(S)$, and, therefore, the points a and b correspond to a single point on S, the cases $Z(U) = \{a, b\}$ and $Z(U) = \{\tilde{x}\}$ for some $\tilde{x} \in (a, b)$ can be actually treated in the same way.

Case 1: *Assume that* $Z(U) = \{\tilde{x}\}$ *for some* $\tilde{x} \in (a, b)$. Since $u(\tilde{x}) = 0$ for every $u \in U$, we consider U as a subspace of periodic functions on $K = [\tilde{x}, \tilde{x} + b - a]$. Assume that U satisfies Property $\mathrm{A_{per}}$. It is then easily seen that U even satisfies Property A on K, i.e., the more general nonperiodic case is given. Indeed, let $u \in U \backslash \{0\}$ and $u^* \in C(K)$ such that $|u^*| = |u|$ on K. Since $u(\tilde{x}) = u(\tilde{x} + b - a) = 0$, it follows that $u^*(\tilde{x}) = u^*(\tilde{x} + b - a) = 0$. Hence, u^* can be continuously extended to a periodic function on $\mathbb{R}$ with period $b - a$, i.e., $u^* \in C_{b-a}$. Then, since U satisfies Property $\mathrm{A_{per}}$, there exists a $\tilde{u} \in U \backslash \{0\}$ for which $\tilde{u} = 0$ a.e. on $Z(u)$ and $\tilde{u}u^* \geqslant 0$ on K.

Thus we have shown that U (as a subspace of $C(K)$) satisfies Property A on K.

But for this case, Pinkus [4, Theorem 4.16] and Li [2] totally classified all $U \subset C(K)$ which satisfy Property A. In particular, they showed that such a subspace U has to have a spline-like structure.

Thus, there still remain the case where $Z(U) = \emptyset$.

Case 2: *Assume that* $Z(U) = \emptyset$. This is the actually interesting case of our periodic approximation problem. We are able to characterize all subspaces U of C_{b-a} which satisfy Property $\mathrm{A_{per}}$.

Before stating the main result, we give the following definition.

Definition 5. We say that $[c, d] \subset \mathbb{R}$ is a *zero interval* of $u \in C_{b-a}$ if $u = 0$ on $[c, d]$, and u does not vanish identically on $(c - \varepsilon, c)$ and on $(d, d + \varepsilon)$ for any $\varepsilon > 0$.

Moreover, we say that zeros $\{x_i\}_{i=1}^{k} \subset \mathbb{R}$ of $u \in C_{b-a}$ such that $x_1 < \cdots < x_k$ are separated zeros of u if there exist $\{y_i\}_{i=1}^{k-1}$ satisfying $y_i \in (x_i, x_{i+1})$, $i = 1, \ldots, k-1$, for which $u(y_i) \neq 0$.

Theorem 5. *Assume that* U *is an* n*-dimensional subspace of* C_{b-a} *satisfying* $Z(U) = \emptyset$. *The following statements* (1) *and* (2) *are equivalent.*

(1) *U satisfies Property* $\mathrm{A_{per}}$.
(2) (a) $[S \backslash Z(u)] \leqslant \dim U(u) = d(u)$ *for every* $u \in U$.
 (b) *For every nonzero* $u \in U$ *and every set* $\{x_i\}_{i=1}^{m+1}$ *of separated zeros of* u *satisfying*

$$a \leqslant x_1 < \cdots < x_m \leqslant b \leqslant x_{m+1} = x_1 + b - a$$

and $x_m - x_1 < b - a$ *where* $1 \leqslant m \leqslant d(u)$ *there exists a* $\tilde{u} \in U(u) \backslash \{0\}$ *such that*

$$(-1)^i \tilde{u}(x) \geqslant 0, \quad x \in [x_i, x_{i+1}], \quad i = 1, \ldots, m.$$

Remark 4. (1) Before proving the theorem, we want to point out that statement (2)(b) is closely related to an important subclass of subspaces, the weak Chebyshev spaces. An m-dimensional subspace V of $C[a,b]$ is said to be a *weak Chebyshev* (WT-) subspace if every $v \in V$ has at most $m-1$ sign changes on $[a,b]$, i.e., there do not exist points $a \leqslant x_1 < \cdots < x_{m+1} \leqslant b$ such that

$$v(x_i)v(x_{i+1}) < 0, \quad i = 1, \ldots, m.$$

The relationship of statement (2)(b) to WT-spaces is based on the following result (for details on WT-spaces cf. [4, p. 204]):

If V is an m-dimensional WT-subspace of $C[a,b]$ and a set of points is given by

$$y_0 = a < y_1 < \cdots < y_k < b = y_{k+1}, \quad k \leqslant m-1,$$

then there exists a $\tilde{v} \in V \backslash \{0\}$ satisfying

$$(-1)^i \tilde{v}(x) \geqslant 0, \quad x \in [y_{i-1}, y_i], \quad i = 1, \ldots, k+1.$$

(2) Another relationship to properties of WT-spaces is given by the following fact: If $U \subset C_{b-a}$ satisfies Property $\mathrm{A}_{\mathrm{per}}$, and for $u \in U$, $[c,d]$ is a zero interval of u with $a \leqslant c < d \leqslant b$, then $U(u)$ satisfies Property $\mathrm{A}_{\mathrm{per}}$ (Remark 2). Moreover, it follows that $U(u)$ is a WT-subspace on $I_c = [c, c+b-a]$. Indeed, suppose there exists a $\tilde{u} \in U(u) \backslash \{0\}$ with at least $d(u)$ sign changes on I_c. This implies that $[I_c \backslash Z(\tilde{u})] \geqslant d(u) + 1$, while in view of Theorem 4,

$$[I_c \backslash Z(\tilde{u})] \leqslant \dim U(\tilde{u}) \leqslant d(u),$$

a contradiction (recall that $\tilde{u}(c) = \tilde{u}(c+b-a) = 0$).

In addition, it follows that Case 1 is given, because $c \in Z(U(u))$. Hence applying the classification results of the nonperiodic case, a characterization of $U(u)$ by a spline-like structure is obtained (see Case 1 above).

Proof of Theorem 5. (1) $\Rightarrow$ (2)(a). This is a consequence of Theorem 4.

(1) $\Rightarrow$ (2)(b). Let $u \in U \backslash \{0\}$ and let for some $m \in \{1, \ldots, d(u)\}$ a set $\{x_i\}_{i=1}^{m+1}$ of separated zeros of u be given satisfying

$$a \leqslant x_1 < x_2 < \cdots < x_m \leqslant b \leqslant x_{m+1} = x_1 + b - a$$

and $x_m - x_1 < b - a$. In particular, $x_{m+1} > x_m$, because $x_{m+1} - x_1 = b - a$. Set $t_i = x_i$, $i = 1, \ldots, m$ and complete this set by points $t_m < t_{m+1} < \cdots < t_{d(u)} < x_{m+1}$ to a set of $d(u)$ points. Let $\{v_1, \ldots, v_{d(u)}\}$ form a basis of $U(u)$. We distinguish.

Assume first that $\det(v_i(t_j))_{i,j=1}^{d(u)} \neq 0$. Then $m < d(u)$, because $u \in U(u)$ and $u(t_i) = 0$, $i = 1, \ldots, m$. Hence there exists a $\hat{u} \in U(u)$ satisfying $\hat{u}(t_i) = 0$, $i = 1, \ldots, d(u) - 1$, and $\hat{u}(t_{d(u)}) = 1$. In particular, $\hat{u}(x_{m+1}) = 0$. Then there exists a $u^* \in C_{b-a}$ such that

$$u^*(x) = \begin{cases} (-1)^i |\hat{u}(x)| & \text{if } x \in [t_i, t_{i+1}],\ i = 1, \ldots, m-1, \\ (-1)^m |\hat{u}(x)| & \text{if } x \in [t_m, x_{m+1}]. \end{cases}$$

Hence it follows that $|u^*| = |\hat{u}|$. Since U satisfies Property A_{per}, there exists a $\tilde{u} \in U(\hat{u})\backslash\{0\} \subset U(u)\backslash\{0\}$ satisfying

$$\tilde{u}u^* \geqslant 0 \quad \text{on } S.$$

This implies that

$$(-1)^i \tilde{u}(x) \geqslant 0, \quad x \in [x_i, x_{i+1}], \ i = 1, \dots, m.$$

If $\det(v_i(t_j))_{i,j=1}^{d(u)} = 0$, there exists a non-zero $\hat{u} \in U(u)$ satisfying $\hat{u}(t_i) = 0$, $i = 1, \dots, d(u)$. Then concluding analogously as above we obtain the desired statement.

(2) $\Rightarrow$ (1): Let $u^* \in U\backslash\{0\}$ and assume that $S\backslash Z(u^*) = \bigcup_{i=1}^{l} A_i$, the union of the connected components. To show Property A_{per} we must prove that for any choice of $\varepsilon_i \in \{-1, 1\}$, $i = 1, \dots, l$ there exists a $\tilde{u} \in U(u^*)\backslash\{0\}$ such that $\varepsilon_i \tilde{u} \geqslant 0$ on A_i, $i = 1, \dots, l$.

Let any set $\{\varepsilon_1, \dots, \varepsilon_l\}$ of signs be given. It first follows from (2)(a) that $l \leqslant \dim U(u^*) = d(u^*)$. If $Z(u^*) = \emptyset$, then $l = 1$ and setting $\tilde{u} = \varepsilon_1 |u^*| \in U(u^*)\backslash\{0\}$, the statement follows. Therefore, assume that $Z(u^*) \neq \emptyset$. Then, there must exist a set $\{x_i\}_{i=1}^{m+1}$ of separated zeros of u^* satisfying

$$a \leqslant x_1 < x_2 < \cdots < x_m \leqslant b \leqslant x_{m+1} = x_1 + b - a$$

and

$$\bigcup_{i=1}^{i_1} A_i \subset (x_1, x_2) \quad \text{if } \varepsilon_1 = \cdots = \varepsilon_{i_1},$$

$$\bigcup_{i=i_1+1}^{i_2} A_i \subset (x_2, x_3) \quad \text{if } \varepsilon_{i_1+1} = -\varepsilon_{i_1}, \varepsilon_{i_1+1} = \cdots = \varepsilon_{i_2},$$

$$\vdots$$

$$\bigcup_{i=i_{m-1}+1}^{i_m} A_i \subset (x_m, x_{m+1}) \quad \text{if } \varepsilon_{i_{m-1}+1} = -\varepsilon_{i_{m-1}}, \varepsilon_{i_{m-1}+1} = \cdots = \varepsilon_{i_m} = \varepsilon_l.$$

Of course, $1 \leqslant m \leqslant l \leqslant d(u^*)$ and $x_m < x_{m+1}$ which implies that $x_m - x_1 < b - a$. Then by hypothesis, we obtain a $\tilde{u} \in U(u^*)\backslash\{0\}$ satisfying

$$(-1)^i \tilde{u}(x) \geqslant 0, \quad x \in [x_i, x_{i+1}], \ i = 1, \dots, m.$$

Assume, without loss of generality, that $\varepsilon_1 = -1$. Then by the choice of $\{x_i\}_{i=1}^{m+1}$, we have

$$\varepsilon_i \tilde{u} \geqslant 0 \quad \text{on } A_i \ i = 1, \dots, l.$$

This completes the proof of Theorem 5. $\square$

Before presenting examples of some nontrivial classes of subspaces which satisfy Property A_{per} we want to point out some differences between the characterizations of Property A in the nonperiodic case due to Pinkus and Li and our characterization of Property A_{per}. For instance, Li [2] gave the following characterization.

Theorem 6. *Let U denote a finite-dimensional subspace of $C[a,b]$ and assume that $Z(U) \cap (a,b) = \emptyset$. Then U satisfies Property* A *if and only if U satisfies the following conditions*:

(1) *U is a weak Chebyshev space*;
(2) *$U([c,d]) = U([a,d]) \oplus U([c,b])$ for all $a < c < d < b$, where for any $a \leqslant \alpha \leqslant \beta \leqslant b$,*

$$U([\alpha,\beta]) = \{u \in U : u = 0 \text{ on } [\alpha,\beta]\}.$$

Remark 5. (1) The second condition implies that every function $u \in U([c,d])$ 'generates' a function v in U such that $v = 0$ on $[a,d]$ and $v = u$ on $[d,b]$ (and, analogously, a function $\tilde{v}$ in U such that $\tilde{v} = 0$ on $[c,b]$ and $\tilde{v} = u$ on $[a,c]$). This property is not true in the periodic case in general: For instance, let $K = [0,3]$ and assume that $U = \mathrm{span}\{u_1, u_2\} \subset C_{3-0}$ where $u_1(x) = 1$ and

$$u_2(x) = \begin{cases} 1 - x & \text{if } 0 \leqslant x \leqslant 1, \\ 0 & \text{if } 1 < x < 2, \\ x - 2 & \text{if } 2 \leqslant x \leqslant 3. \end{cases}$$

Then U satisfies Property $\mathrm{A_{per}}$, because U is a space of piecewise polynomials on K with the knots $x_i = i$, $i = 0,1,2,3$ (see Example 3). But, U fails to satisfy statement (2) of Theorem 6, since $U([1,2]) = \mathrm{span}\{u_2\}$ and $U([0,2]) = \{0\}$, $U([1,3]) = \{0\}$.

(2) The above example fails to be a weak Chebyshev space, because $u_2 - \frac{1}{2}u_1$ has two sign changes in $(0,3)$. Hence statement (1) of Theorem 6 is also not true in the periodic case in general.

Example 2 (Trigonometric polynomials). Let $K = [0,2\pi]$ and assume that U denotes the $(2n+1)$-dimensional subspace of all trigonometric polynomials u of order n, i.e.,

$$u(x) = a_0 + \sum_{j=1}^{n} (a_j \cos jx + b_j \sin jx), \quad x \in [0,2\pi].$$

It is well-known that U is a *Haar system* on $[0,2\pi)$, i.e., every nonzero $u \in U$ has at most $2n$ zeros in $[0,2\pi)$. Hence $U(u) = U$ for every nonzero $u \in U$ and $[K \backslash Z(u)] \leqslant 2n+1 = \dim U$ which implies, in view of Remark 1, that U satisfies Property A on K. Then in particular, U satisfies Property $\mathrm{A_{per}}$.

Example 3 (Piecing together Haar systems). Let $a = e_0 < e_1 < \cdots < e_{k+1} = b$. On each interval $I_i = [e_{i-1}, e_i]$, let U_i be a Haar system of real-valued continuous functions with dimension $n_i \geqslant 1$, $i = 1,\ldots,k+1$. For convenience, we especially assume that $n_1 \geqslant 2$ and $n_{k+1} \geqslant 2$. V will denote the subspace of $C[a,b]$ defined by

$$V = \{v \in C[a,b] : v|_{I_i} \in U_i, \ i = 1,\ldots,k+1\}.$$

It is well-known (cf. [4, p. 80]) that $\dim V = \sum_{i=1}^{k+1} n_i - k$ and V is a WT-system on $[a,b]$. Moreover, V satisfies Property A on $[a,b]$.

To investigate its periodic analogue we consider the subspace U of C_{b-a} defined by

$$U = \{u \in C_{b-a} : u|_{I_i} \in U_i,\ \ i = 1, \ldots, k+1\}. \tag{4.2}$$

Thus, U is the space of all periodic extensions of functions $v \in V$ such that $v(a) = v(b)$.

Theorem 7. *Let U be the space of periodic functions defined in* (4.2). *Then U satisfies Property* $\mathrm{A}_{\mathrm{per}}$.

To apply Theorem 5 we divide the proof of Theorem 7 into several parts.

Claim 3.1. *Let U and V be given as above. Then*

$$\dim U = \dim V - 1. \tag{4.3}$$

Proof. Since $n_1 \geqslant 2$ and $n_{k+1} \geqslant 2$, by the Haar condition of U_1 and U_{k+1}, respectively, there exists a $v \in V$ such that $v(a) = 1$, $v(e_1) = 0$, $v = 0$ on (e_1, e_k), and $v(e_k) = 0$, $v(b) = -1$. Hence v cannot periodically extended to a function in U which implies that

$$\dim U < \dim V.$$

To show the statement we set $\dim V = n+1$ $(n \geqslant 0)$ and suppose that $\{v_1, \ldots, v_{n+1}\}$ forms a basis of V such that

$$v_1 > 0 \text{ on } \quad [a,b],\ \ v_i(a) = 0, \quad i = 2, \ldots, n+1.$$

(Recall that each U_i is a Haar system on I_i which implies that there exists a positive function in U_i.)

We show that $v_i(b) \neq 0$ for some $i \in \{2, \ldots, n+1\}$. On the contrary, assume that $v_i(b) = 0,\ i = 2, \ldots, n+1$. Let $u \in V$ such that $u(a) = 1$, $u(e_1) = 0$, $u = 0$ on (e_1, e_k), and $u(e_k) = 0$, $u(b) = 1$ (u can be found analogously as the function v above using the Haar condition of U_1 and U_{k+1}, respectively). Hence it follows that $\{u, v_2, \ldots, v_{n+1}\}$ are linearly independent and can be periodically extended to functions in U. This implies $\dim U \geqslant n+1 = \dim V$, a contradiction. Thus it follows that $v_l(b) \neq 0$ for some $l \in \{2, \ldots, n+1\}$. Consider the n linearly independent functions in V,

$$\tilde{v}_1 = v_1 + \frac{v_1(a) - v_1(b)}{v_l(b)} v_l, \quad \tilde{v}_i = v_i - \frac{v_i(b)}{v_l(b)} v_l, \quad i = 2, \ldots, n+1,\ \ i \neq l.$$

Then we obtain that

$$0 \neq \tilde{v}_1(a) = v_1(a) = \tilde{v}_1(b), \quad \tilde{v}_i(a) = \tilde{v}_i(b) = 0, \quad i = 2, \ldots, n+1,\ \ i \neq l$$

which implies that $\{\tilde{v}_1, \ldots, \tilde{v}_{l-1}, \tilde{v}_{l+1}, \ldots, \tilde{v}_{n+1}\}$ can be periodically extended to functions in U. Thus,

$$n \leqslant \dim U < \dim V = n+1,$$

and it follows that $\dim U = n = \dim V - 1$. $\square$

Claim 3.2. $Z(U) = \emptyset$.

Proof. Since each U_i is a Haar system on I_i, there exists a positive function in U_i, $i = 1, \dots, k+1$. Then piecing together such functions we obtain a continuous and positive function $\tilde{v}$ on $[a, e_k]$ such that $\tilde{v}|_{I_i} \in U_i$, $i = 1, \dots, k$. Since U_{k+1} is a Haar system on I_{k+1} and $n_{k+1} \geqslant 2$, by interpolation we construct a function $\hat{v} \in U_{k+1}$ such that $\hat{v}(e_k) = \tilde{v}(e_k)$ and $\hat{v}(b) = \tilde{v}(a)$. Piecing together $\hat{v}$ and $\tilde{v}$ we then obtain a function $\tilde{u} \in U$ such that $\tilde{u} > 0$ on $[a, e_k]$. This implies that $[a, e_k] \cap Z(U) = \emptyset$. Analogously, we find a function $\hat{u} \in U$ such that $\hat{u} > 0$ on $[e_1, b]$ which implies that $[e_1, b] \cap Z(U) = \emptyset$.
Thus the statement is proved. $\square$

Claim 3.3. *Let $u \in U$ and assume that $\tilde{u} \in V$ such that $\tilde{u} = u|_{[a,b]}$. Then we obtain*

$$d(u) = \dim U(u) = \begin{cases} \dim V(\tilde{u}) & \text{if } I_1 \cup I_{k+1} \subset Z(u), \\ \dim V(\tilde{u}) - 1 & \text{otherwise}. \end{cases} \tag{4.4}$$

Proof. It is obvious that $d(u) = \dim U(u) \leqslant \dim V(\tilde{u})$. Assume first that $I_1 \cup I_{k+1} \subset Z(u)$. Then, if $v \in V(\tilde{u})$, it follows that $v = 0$ on $I_1 \cup I_{k+1}$. Hence v has a periodic extension in $U(u)$ which implies that $\dim V(\tilde{u}) \leqslant d(u)$.

Assume now, without loss of generality, that u does not vanish identically on I_{k+1}. Using the Haar condition of U_{k+1} we find a $v \in V$ such that

$$v = 0 \text{ on } \quad [a, e_k], \quad v(b) = 1.$$

Since I_{k+1} fails to be a subset of $Z(\tilde{u})$, it follows that $v \in V(\tilde{u})$. But, since v has no periodic extension in U, we obtain that

$$d(u) < \dim V(\tilde{u}).$$

Arguing similarly as in the proof of (4.3), we can then show that

$$d(u) = \dim V(\tilde{u}) - 1. \qquad \square$$

Claim 3.4. *Let $u \in U$. Then*

$$[S \backslash Z(u)] \leqslant d(u). \tag{4.5}$$

Proof. Set $\tilde{u} = u|_{[a,b]}$. Then $\tilde{u} \in V$. Since V satisfies Property A, following Theorem 2 and Remark 1 we obtain that

$$[[a, b] \backslash Z(\tilde{u})] \leqslant \dim V(\tilde{u}).$$

This implies that if $u(a) = u(b) \neq 0$,

$$[S \backslash Z(u)] \leqslant \dim V(\tilde{u}) - 1,$$

because the first and the last component of $[a, b] \backslash Z(\tilde{u})$ reduce to one connected component of $S \backslash Z(u)$. Hence in view of (4.4), the statement follows. Moreover, the statement also follows, if $\dim V(\tilde{u}) = d(u)$.

Thus we have still to consider the case when $u(a) = u(b) = 0$ and $d(u) = \dim V(\tilde{u}) - 1$. In view of (4.4), let us assume that I_{k+1} fails to be a subset of $Z(\tilde{u})$. Moreover, suppose that

$$[[a,b] \backslash Z(\tilde{u})] = \dim V(\tilde{u}) = d(u) + 1.$$

Since $\tilde{u}$ does not vanish identically on I_{k+1} and $\tilde{u}(b) = 0$, it has exactly $0 \leqslant r \leqslant n_{k+1} - 2$ zeros $e_k \leqslant z_1 < \cdots < z_r < b$ there (recall that U_{k+1} is a Haar system on I_{k+1}). Assume that $\tilde{u} > 0$ on $(b - \varepsilon, b)$ for some $\varepsilon > 0$. Interpolating by U_{k+1} on I_{k+1} we obtain a function $\tilde{v} \in V$ such that

$$\tilde{v} = 0 \text{ on } [a, e_k], \quad \tilde{v}(z_i) = 0, \ i = 1, \ldots, r, \quad \tilde{v}(b) = 1.$$

Then for some sufficiently small $c > 0$, the function $\tilde{u} - c\tilde{v} \in V$ has a sign change on $(b - \varepsilon, b)$ which implies that

$$[[a,b] \backslash Z(\tilde{u} - c\tilde{v})] \geqslant \dim V(\tilde{u}) + 1.$$

Moreover, since $\tilde{v} = 0$ on $[a, e_k]$ and $\tilde{u} - c\tilde{v}$ does not vanish identically on I_{k+1}, we obviously obtain that

$$V(\tilde{u} - c\tilde{v}) = V(\tilde{u}).$$

Thus it follows that

$$[[a,b] \backslash Z(\tilde{u} - c\tilde{v})] \geqslant \dim V(\tilde{u} - c\tilde{v}) + 1,$$

in contradiction to Property A of V.

Thus, we have shown that in the case when $u(a) = u(b) = 0$ and $d(u) = \dim V(\tilde{u}) - 1$,

$$[S \backslash Z(u)] = [[a,b] \backslash Z(\tilde{u})] \leqslant \dim V(\tilde{u}) - 1 = d(u). \qquad \square$$

Claim 3.5. *U is a WT-system on $[a,b]$, if n ($= \dim U$) is odd.*

Proof. Assume that there exists a $\hat{u} \in U$ such that $\hat{u}$ has at least n sign changes in (a,b). If $\hat{u}(a) = \hat{u}(b) = 0$, then

$$[[a,b] \backslash Z(\hat{u})] = [S \backslash Z(\hat{u})] \geqslant n + 1 \geqslant d(\hat{u}) + 1,$$

which contradicts (4.5). Hence, $\hat{u}(a) = \hat{u}(b) \neq 0$. But then, in view of the fact that n is odd, $\hat{u}$ must have at least $n + 1$ sign changes in (a,b) which contradicts the property of V to be a WT-system on $[a,b]$. $\square$

Claim 3.6. *Set*

$$\tilde{U} = \{u \in U : u(a) = 0\}.$$

Then $\dim \tilde{U} = n - 1$ *and $\tilde{U}$ is a WT-system on $[a,b]$.*

Proof. Since there exists a $u \in U$ such that

$$u(a) = 1, \quad u = 0 \text{ on } [e_1, e_k], \quad u(b) = 1,$$

it is easily seen that $\dim \tilde{U} = n-1$. Assume now that there exists a $\tilde{u} \in \tilde{U}$ with at least $n-1$ sign changes in (a,b). Of course, $\tilde{u}(a) = \tilde{u}(b) = 0$. Then it follows from (4.5) that $[a,b] \backslash Z(\tilde{u}) = \bigcup_{i=1}^{l} A_i$ (the union of the connected components) with $l = n$, which implies that $\tilde{u}$ has exactly $n-1$ sign changes. Consider first the case when n is even. Then $\tilde{u}$ has different sign on A_1 and on A_l, respectively. Let $\tilde{v} \in V$ such that $\tilde{v}(a)\tilde{v}(b) < 0$ (this function can be found analogously as the function v defined in the proof of (4.3)). Then for some sufficiently small ε the function $\tilde{u} + \varepsilon\tilde{v} \in V$ has at least $n+1$ sign changes in (a,b) which contradicts the property of V to be a WT-system on $[a,b]$.

The case when n is odd can be treated analogously using a function $\tilde{v} \in V$ such that $\tilde{v}(a)\tilde{v}(b) > 0$.

Thus it follows that $\tilde{U}$ is a WT-system on $[a,b]$. □

Claim 3.7. *Let $u \in U \backslash \{0\}$ and let a set $\{x_i\}_{i=1}^{m+1}$ of separated zeros of u be given satisfying*

$$a \leqslant x_1 < x_2 < \cdots < x_m \leqslant b \leqslant x_{m+1} = x_1 + b - a \tag{4.6}$$

and $x_m - x_1 < b - a$ where $1 \leqslant m \leqslant d(u)$. Then there exists a $\tilde{u} \in U(u) \backslash \{0\}$ such that

$$(-1)^i \tilde{u}(x) \geqslant 0, \quad x \in [x_i, x_{i+1}], \ i = 1, \ldots, m.$$

Proof. We prove the statement by considering several cases.

Case 3.6.1: *Assume that $U(u) = U$.* Let a set of separated zeros of u be given by (4.6). Suppose first that $m = n$. Since each $v \in V$ has at most $n_i - 1$ separated zeros in I_i, $i = 1, \ldots, k+1$, and $n + 1 = \dim V = \sum_{i=1}^{k+1} n_i - k$, it follows that each $v \in V$ has at most n separated zeros in $[a,b]$. Hence the assumption $m = n$ implies that $Z(u) \cap [x_1, x_{m+1}] = \{x_i\}_{i=1}^{m+1}$ and u has exactly $n_i - 1$ of its zeros in each I_i, $i = 1, \ldots, k+1$. Moreover, $e_1 \notin Z(u)$, because otherwise e_1 is a common zero of $u|_{I_1}$ and $u|_{I_2}$ which implies that u would have at most $(n_1 - 1) + (n_2 - 1) - 1$ zeros in $I_1 \cup I_2$. Then u could have at most $\sum_{i=1}^{k+1} n_i - (k+1) - 1 = n - 1$ zeros in $[a,b]$ contradicting $m = n$. Analogously we obtain that $Z(u) \cap \{e_i\}_{i=1}^{k} = \emptyset$. Then, since each U_i is a Haar system on I_i, all the zeros of u in (a,b) must be sign changes.

We distinguish: If $x_m < b$, then u changes the sign at $\{x_i\}_{i=2}^{m}$ and setting $\tilde{u} = \varepsilon u$ for some $\varepsilon \in \{-1, 1\}$ the statement follows.

If $x_m = b$, then $x_1 > a$, because $x_m - x_1 < b - a$. Moreover, $u(a) = 0$, since $u(x_m) = 0$ and $u \in C_{b-a}$. Then u would have $m + 1 = n + 1$ separated zeros $\{a, x_1, \ldots, x_m\}$ in $[a,b]$ contradicting the above arguments on V.

Suppose now that $m \leqslant n - 1$ and n is even. Set

$$y_0 = a, \ y_i = x_i, \ i = 1, \ldots, m, \ y_{m+1} = b \quad \text{if } m \text{ is even}$$

(then, in fact, $m \leqslant n - 2$, because n is even), and

$$y_0 = a, \ y_i = x_{i+1}, \ i = 1, \ldots, m-1, \ y_m = b \quad \text{if } m \text{ is odd}.$$

In both cases, using the statements on WT-systems given in Remark 4 we find a $\tilde{u}\in\tilde{U}\backslash\{0\}$ (recall that we have shown in Claim 3.6 that $\tilde{U}$ is a WT-system on $[a,b]$) such that

$$(-1)^i\tilde{u}(x)\geqslant 0,\ \ x\in[y_i,y_{i+1}],\ \ i=0,\dots,m \quad \text{if } m \text{ is even},$$

$$(-1)^{i+1}\tilde{u}(x)\geqslant 0,\ \ x\in[y_i,y_{i+1}],\ \ i=0,\dots,m-1 \quad \text{if } m \text{ is odd}.$$

(If $x_1=a$ or $x_m=b$, the inequalities are also true for the degenerate intervals $[y_0,y_1]$, $[y_{m-1},y_m]$ or $[y_m,y_{m+1}]$, respectively, because $\bar{u}(a)=\bar{u}(b)=0$ for every $\bar{u}\in\tilde{U}$.)

Thus in both cases it follows that

$$(-1)^m\tilde{u}(x)\geqslant 0, \quad x\in[a,x_1]\cup[x_m,b],$$

which corresponds to the sign behavior of $\tilde{u}$ on $[x_m,x_{m+1}]$. Hence $\tilde{u}$ has the desired properties.

Suppose now that n is odd and $m\leqslant n-1$. Then by Claim 3.5, U itself is a WT-system on $[a,b]$. Replacing the subspace $\tilde{U}$ by U, if necessary, and arguing analogously as above we obtain a function $\tilde{u}\in U\backslash\{0\}$ with the desired properties.

Case 3.6.2: *Assume that u has at least two zero intervals* $J_1=[e_j,e_l]$ *and* $J_2=[e_p,e_q]$ *in* $[a,b]$ *such that* $e_l<e_p$, *and at most finitely many zeros in* $[e_l,e_p]$. Let $\{x_i\}_{i=1}^m\cap(e_l,e_p)=\{y_i\}_{i=1}^r$ such that $y_0=e_l<y_1<\cdots<y_r<e_p=y_{r+1}$. Define $\hat{u}\in V$ satisfying $\hat{u}=0$ on $[a,e_l]\cup[e_p,b]$ and $\hat{u}=u$ on (e_l,e_p). Since V satisfies Property A, the subspace $V(\hat{u})$ satisfies Property A. To use this property we distinguish several cases:

Consider first the cases when $x_1\notin(e_l,e_p)$ and $x_1\in(e_l,e_p)$, m even, respectively (hence $x_1=y_1$ in the second case). In both cases, by the Property A there exists a $\tilde{u}\in V(\hat{u})\backslash\{0\}$ such that

$$(-1)^i\tilde{u}(x)\geqslant 0, \quad x\in[y_i,y_{i+1}],\ \ i=0,\dots,r.$$

Finally assume that $x_1\in(e_l,e_p)$ and m is odd. Define $\tilde{y}_0=e_l$ and $\tilde{y}_i=y_{i+1},\ i=1,\dots,r$. By the Property A there exists a $\tilde{u}\in V(\hat{u})\backslash\{0\}$ such that

$$(-1)^{i+1}\tilde{u}(x)\geqslant 0, \quad x\in[\tilde{y}_i,\tilde{y}_{i+1}],\ \ i=0,\dots,r-1.$$

Moreover, in all cases $\tilde{u}(a)=\tilde{u}(b)=0$ which implies that $\tilde{u}$ has a periodic extension in C_{b-a} (again denoted by $\tilde{u}$). Therefore, $\tilde{u}\in U(u)$ and $\varepsilon\tilde{u}$ has the desired properties for some $\varepsilon\in\{-1,1\}$.

Case 3.6.3: *Assume that u has a unique zero interval* $J=[e_p,e_q]$ *in* $[a,b]$. To derive this case from Case 3.6.2 we generate a subspace $\tilde{V}$ of piecing together Haar systems for a bigger knot sequence as follows:

Let

$$e_{k+1+i}=e_i+b-a, \quad I_{k+1+i}=[e_{k+i},e_{k+1+i}]$$

and

$$U_{k+1+i}=\{u\in C(I_{k+1+i}):u(x)=\tilde{u}(x-(b-a)),\ \ x\in I_{k+1+i},\ \text{for some } \tilde{u}\in U_i\},$$

$i = 1, \dots, k+1$. We consider the linear space $\tilde{V}$ defined by

$$\tilde{V} = \{v \in C[e_0, e_{2k+2}] : v|_{I_i} \in U_i,\ i = 1, \dots, 2k+2\}.$$

Of course, $\tilde{V}$ has the same properties as V. In particular, it satisfies Property A. Moreover, the given subspace U of C_{b-a} can also be defined by

$$U = \{u \in C_{b-a} : u|_{I_i} \in U_i,\ i = l, \dots, l+k\}$$

for any $l \in \{1, \dots, k+2\}$. We now consider the given function u on $[e_p, e_{q+k+1}]$. Then by hypothesis, $u = 0$ on $[e_p, e_q] \cup [e_{p+k+1}, e_{q+k+1}]$ and u has at most finitely many zeros in $[e_q, e_{p+k+1}]$. As in Case 3.6.2 we define $\hat{u} \in \tilde{V}$ satisfying $\hat{u} = 0$ on $[e_0, e_q] \cup [e_{p+k+1}, e_{2k+2}]$ and $\hat{u} = u$ on (e_q, e_{p+k+1}). Since $\tilde{u}|_{I_i} \in U_i,\ i = p+1, \dots, p+k+1$, and $\tilde{u}(e_p) = \tilde{u}(e_{p+k+1}) = 0$ for every $\tilde{u} \in \tilde{V}(\hat{u})$, every function $\tilde{u}|_{[e_p, e_{p+k+1}]}$ has a periodic extension in U. Moreover, since $\tilde{V}(\hat{u})$ satisfies Property A, similarly arguing as in Case 3.6.2 we find a $\tilde{u} \in \tilde{V}(\hat{u})$ such that $\tilde{u}$ has the desired sign behaviour on $[e_p, e_{p+k+1}]$ (where the separated zeros $\{x_i\}_{i=1}^{m+1}$ are identified with a subset of the sphere and, therefore, they correspond to a set of separated zeros in $[e_p, e_{p+k+1}]$). Thus the extension of $\tilde{u}$ in U (again denoted by $\tilde{u}$) is a function with the desired properties on $[e_p, e_{p+k+1}]$ and, therefore, on $[a, b]$.

This completes the proof of Claim 3.7. □

Proof of Theorem 7. From Claim 3.2 it follows that $Z(U) = \emptyset$. Moreover, in view of Claim 3.4, statement (2)(a) of Theorem 5 is satisfied. Finally, statement (2)(b) of Theorem 5 follows from Claim 3.7.

Hence by Theorem 5, U satisfies Property A_{per}. □

Example 4 (Periodic splines). Given $k \geqslant 0$ and $l \geqslant 1$, let $a = e_0 < e_1 < \cdots < e_{k+1} = b$. Extend this knot vector to a knot sequence on $\mathbb{R}$ by

$$e_{i+j(k+1)} = e_i + j(b-a), \quad i = 0, \dots, k+1,\ j \in \mathbb{Z} \backslash \{0\}.$$

Set $\Delta = \{e_i\}_{i \in \mathbb{Z}}$ and $I_i = [e_{i-1}, e_i]$, $i \in \mathbb{Z}$. By Π_l we denote the linear space of all polynomials of degree at most l. For any $q \in \{1, \dots, l\}$ we consider the linear space $S_l^{l-q}(\Delta)$ defined by

$$S_l^{l-q}(\Delta) = \{s \in C^{l-q}(\mathbb{R}) : s|_{I_i} \in \Pi_l, i \in \mathbb{Z}\},$$

the subspace of *polynomial spline functions* of degree l with the fixed knots $\{e_i\}_{i \in \mathbb{Z}}$ of multiplicity q. It is well-known (cf. [6, Theorem 4.5]) that $\dim S_l^{l-q}(\Delta)|_{[a,b]} = l + 1 + qk$ and a natural basis on $[a, b]$ is given by

$$1, x, \dots, x^l, (x-e_1)_+^l, \dots, (x-e_1)_+^{l-q+1}, \dots, (x-e_k)_+^l, \dots, (x-e_k)_+^{l-q+1},$$

where

$$(x-e_i)_+^r := \begin{cases} (x-e_i)^r & \text{if } x \geqslant e_i, \\ 0 & \text{if } x < e_i. \end{cases}$$

Moreover, it is well-known that $S_l^{l-q}(\Delta)$ is a WT-system on $[a,b]$ [6, Theorem 4.55] and satisfies Property A there [4, p. 81].

For that what follows we need a local basis of $S_l^{l-q}(\Delta)$, the basis of B-splines. To define it we split up each knot e_i according to its multiplicity q by setting

$$e_i = y_{iq} = y_{iq+1} = \cdots = y_{(i+1)q-1}, \quad i \in \mathbb{Z}.$$

Then it is well-known [6, Theorem 4.9] that a basis of $S_l^{l-q}(\Delta)$ is given by $\{B_\mu\}_{\mu\in\mathbb{Z}}$ where B_μ is the unique *B-spline* satisfying

$$B_\mu = 0 \quad \text{on } \mathbb{R}\backslash(y_\mu, y_{\mu+l+1}),$$

$$B_\mu(x) > 0 \quad \text{for } x \in (y_\mu, y_{\mu+l+1}),$$

$$\sum_{\mu\in\mathbb{Z}} B_\mu(x) = 1 \quad \text{for } x \in \mathbb{R}.$$

Moreover, it is well-known [6, Theorem 4.64] that every subsystem $\{B_{\mu_1}, B_{\mu_1+1}, \ldots, B_{\mu_2}\}$ where $\mu_1, \mu_2 \in \mathbb{Z}$, $\mu_1 < \mu_2$, spans a WT-space.

We are now interested in the subspace

$$P_l^{l-q}(\Delta) = S_l^{l-q}(\Delta) \cap C_{b-a}, \tag{4.7}$$

the subspace of *periodic splines* of degree l with the fixed knots $\{e_i\}_{i\in\mathbb{Z}}$ of multiplicity q. It is easily verified that

$$\dim P_l^{l-q}(\Delta) = l + 1 + qk - (l - q + 1) = q(k+1).$$

Theorem 8. *Let* $U = P_l^{l-q}(\Delta)$, *the space of periodic splines defined in* (4.7). *Then* U *satisfies Property* $\mathrm{A_{per}}$.

To prove this statement we distinguish two cases.

Case 4.1: *Let* $q = l$. Then $S_l^0(\Delta)|_{[a,b]}$ is obviously a space of piecing together the Haar systems $U_i = \Pi_l, i = 1, \ldots, k+1$. This implies that $S_l^0(\Delta)|_{[a,b]}$ corresponds to a space V as considered in Example 3. Hence by the arguments in the proof of Example 3, the space $U = P_l^0(\Delta)$ satisfies Property $\mathrm{A_{per}}$.

Case 4.2: *Assume that* $q \in \{1, \ldots, l-1\}$. To show that $U = P_l^{l-q}(\Delta)$ satisfies Property $\mathrm{A_{per}}$ we divide the proof into several parts.

Claim 4.1. *Let* $u \in U$. *Then*

$$[S\backslash Z(u)] \leqslant \dim U(u). \tag{4.8}$$

Proof. Assume first that $U(u) = U$ and $S\backslash Z(u) = \bigcup_{i=1}^r A_i$, the union of the connected components, where $r \geqslant \dim U + 1 = q(k+1) + 1$. It is easily seen that for each $i \in \{1, \ldots, r\}$ there exists a $z_i \in A_i$ such that $u'(z_i) = 0$ (the derivative of u) and $\{z_i\}_{i=1}^{r+1}$ is a set of separated zeros of u' (as a subset of $\mathbb{R}$) satisfying, without loss of generality,

$$a \leqslant z_1 < \cdots < z_r \leqslant b \leqslant z_{r+1} = z_1 + b - a$$

and $z_r - z_1 < b - a$. Hence

$$[S \backslash Z(u')] \geqslant r.$$

By a repeated application of this argument we finally obtain that

$$[S \backslash Z(u^{(l-q)})] \geqslant r.$$

Moreover, $u^{(l-q)}$ is a continuous and periodic spline function of degree q which implies that

$$u^{(l-q)} \in P_q^0(\Delta).$$

Since $\dim P_q^0(\Delta) = q(k+1)$, we then have got that

$$[S \backslash Z(u^{(l-q)})] \geqslant r \geqslant q(k+1) + 1 = \dim P_q^0(\Delta) + 1.$$

But this contradicts (4.5), because $P_q^0(\Delta)$ is a space of piecing together the Haar systems $U_i = \Pi_q, i \in \mathbb{Z}$, as considered in Example 3.

Assume now that u has a unique zero interval $J = [e_\mu, e_v]$ in $[a, b]$. To determine the dimension of $U(u)$ we consider the interval $\tilde{J} = [e_\mu, e_{v+k+1}]$. Then u has the unique zero intervals J and $\hat{J} = [e_{\mu+k+1}, e_{v+k+1}]$ in $\tilde{J}$, and, since

$$e_v = y_{vq+i}, \quad e_{\mu+k+1} = y_{(\mu+k+1)q+i}, \quad i = 0, \dots, q-1,$$

it is easily verified that

$$U(u)|_{\tilde{J}} = \operatorname{span}\{B_{vq}, B_{vq+1}, \dots, B_{(\mu+k+2)q-l-2}\}|_{\tilde{J}}.$$

Therefore,

$$\tilde{d} := \dim U(u) = \dim U(u)|_{\tilde{J}} = (\mu + k + 2 - v)q - l - 1.$$

Suppose that $[S \backslash Z(u)] \geqslant \tilde{d} + 1$. Then, since $u(e_v) = u(e_{\mu+k+1}) = 0$, u has at least $\tilde{d} + 2$ separated zeros

$$z_0 = e_v < z_1 < \cdots < z_{\tilde{d}} < e_{\mu+k+1} = z_{\tilde{d}+1}.$$

Since $u^{(j)}(e_v) = u^{(j)}(e_{\mu+k+1}) = 0, j = 0, \dots, l-q$, it then follows that $u^{(j)}$ has at least $\tilde{d} + j + 2$ separated zeros in $\tilde{J}$, $j = 0, \dots, l-q$. But, $u^{(l-q)} \in P_q^0(\Delta)$ which implies that $u^{(l-q)}$ has at most q separated zeros in each I_i, $i = v+1, \dots, \mu+k+1$, i.e., $u^{(l-q)}$ has at most

$$\sum_{i=v+1}^{\mu+k+1} q = (\mu + k + 1 - v)q < (\mu + k + 1 - v)q + 1 = \tilde{d} + l - q + 2,$$

a contradiction. (This part can also be proved by applying [6, Theorem 4.53].)

Finally, assume that u has exactly r zero intervals $J_i = [e_{\mu_i}, e_{v_i}]$ satisfying $e_{v_i} < e_{\mu_{i+1}}$, $i = 1, \dots, r-1$ with $r \geqslant 2$ in $[a, b]$. Set $J_{r+1} = [e_{\mu_{r+1}}, e_{v_{r+1}}]$ where $e_{\mu_{r+1}} = e_{\mu_1+k+1}$, $e_{v_{r+1}} = e_{v_1+k+1}$, and $\tilde{J}_i = [e_{\mu_i}, e_{v_{i+1}}]$, $i = 1, \dots, r$. Then analogously as above it is easy to see that

$$U(u)|_{\tilde{J}_i} = \text{span}\{B_{\nu_i q}, B_{\nu_i q+1}, \dots, B_{(\mu_{i+1}+1)q-l-2}\}|_{\tilde{J}_i},$$

$i = 1, \dots, r$, and

$$\dim U(u) = \sum_{i=1}^{r} \dim U(u)|_{\tilde{J}_i}. \tag{4.9}$$

Since every $\tilde{J}_i$ corresponds to the interval $\tilde{J}$ in the above considered case of a unique zero interval, we can apply the above arguments and obtain that

$$[\tilde{J}_i \backslash Z(u)] \leqslant \dim U(u)|_{\tilde{J}_i},$$

$i = 1, \dots, r$. Then the statement follows from (4.9).
This completes the proof of Claim 4.1. □

Claim 4.2. *$U = P_l^{l-q}(\Delta)$ is a WT-system on $[a, b]$, if its dimension is odd.*

Proof. Assume that there exists a $\hat{u} \in U$ such that $\hat{u}$ has at least $q(k+1)$ sign changes in (a, b). Then, similarly arguing as in the proof of Claim 4.1, we obtain that $\hat{u}^{(l-q)}$ has at least $q(k+1)$ sign changes in S. In fact, $\hat{u}^{(l-q)}$ must have at least $q(k+1)+1$ sign changes in S, because $q(k+1)$ is odd. Thus it follows that

$$[S \backslash Z(\hat{u}^{(l-q)})] \geqslant q(k+1) + 1.$$

But this contradicts (4.5), since $\hat{u}^{(l-q)} \in P_q^0(\Delta)$ and $P_q^0(\Delta)$ is a space of piecing together the Haar systems $U_i = \Pi_q$, $i \in \mathbb{Z}$, satisfying $\dim P_q^0(\Delta) = q(k+1)$. □

Claim 4.3. *Let $q(k+1)$ be even and define*

$$\tilde{U} = \{\tilde{u} \in U : \tilde{u} \in C^{l-q+1}(e_k - \varepsilon, e_k + \varepsilon) \text{ for } \varepsilon > 0 \text{ sufficiently small}\}.$$

Then $\dim \tilde{U} = q(k+1) - 1$ and $\tilde{U}$ is a WT-system on $[a, b]$.

Proof. Recall first that $u \in C^{l-q}(\mathbb{R})$ for every $u \in U$. Since in addition every $\tilde{u} \in \tilde{U}$ is at least $l - q + 1$ times continuously differentiable in a neighborhood of the knot e_k, the periodic spline space $\tilde{U}$ is defined by the given knot sequence Δ with the difference that e_k (and all of its periodic analogues $\{e_{k+i(k+1)}\}_{i \in \mathbb{Z}}$) are chosen to be of multiplicity $q - 1$ (the multiplicity q of the other knots in Δ remains unchanged). Thus it follows that

$$\tilde{d} := \dim \tilde{U} = \dim U - 1 = q(k+1) - 1.$$

Suppose now that there exists a $\tilde{u} \in \tilde{U}$ such that $\tilde{u}$ has at least $\tilde{d}$ sign changes in (a, b). Since by assumption $\tilde{d}$ is odd, as in the proof of Claim 4.2 we can show that $\tilde{u}^{(l-q)}$ has at least $\tilde{d} + 1$ sign changes in S.

Let $D = \bigcup_{i\in\mathbb{Z}} (e_i, e_{i+1}) \cup \{e_{k+j(k+1)}\}_{j\in\mathbb{Z}}$ and set

$$\hat{u}(x) = \begin{cases} \frac{d}{dx}\tilde{u}^{(l-q)}(x) & \text{if } x\in D,\\ 0 & \text{if } x\in\mathbb{R}\backslash D. \end{cases}$$

Since $\hat{u}$ is a piecewise polynomial of degree $q-1$ on D, it has at most $q-1$ zeros with a sign change in each (e_i, e_{i+1}), $i = 0, \dots, k-2$, and at most $2q-2$ zeros with a sign change in (e_{k-1}, e_{k+1}) (note that $\hat{u}$ is continuous at e_k). Moreover, $\hat{u}$ can change the sign in $(e_i - \delta, e_i + \delta)$, $i = 0, \dots, k-1$, for some $\delta > 0$ sufficiently small. Thus it follows that $\hat{u}$ has at most

$$(k-1)(q-1) + 2q - 2 + k = q(k+1) - 1 = \tilde{d}$$

sign changes in S. On the other hand, $\tilde{u}^{(l-q)}$ has at least $\tilde{d}+1$ sign changes in S which implies that $\hat{u}$ must have at least $\tilde{d}+1$ sign changes in S, a contradiction.

Thus we have shown that $\tilde{U}$ is a WT-system on $[a,b]$. □

Claim 4.4. *Let $u\in U\backslash\{0\}$ and let a set $\{x_i\}_{i=1}^{m+1}$ of separated zeros of u be given satisfying*

$$a \leqslant x_1 < x_2 < \dots < x_m \leqslant b \leqslant x_{m+1} = x_1 + b - a \tag{4.10}$$

and $x_m - x_1 < b - a$ where $1 \leqslant m \leqslant \dim U(u)$. Then there exists a $\tilde{u}\in U(u)\backslash\{0\}$ such that

$$(-1)^i \tilde{u}(x) \geqslant 0, \quad x\in[x_i, x_{i+1}], \; i = 1, \dots, m. \tag{4.11}$$

Proof. We consider several cases.

Case 4.4.1. *Assume that $U(u) = U$.* Let $d := \dim U = q(k+1)$ and let a set of separated zeros of u be given by (4.10). Suppose first that $m = d$. Since $U(u) = U$, u has at most finitely many zeros in $[a,b]$. We show that u changes the sign at x_i, $i = 2, \dots, m$. Otherwise, $u'(x_{i_0}) = 0$ for some $i_0 \in \{2, \dots, m\}$. Moreover, it is easy to see that for every $i\in\{2, \dots, m+1\}$ there exists a $z_i \in (x_{i-1}, x_i)$ such that $u'(z_i) = 0$ and $\{x_{i_0}, z_2, \dots, z_{m+1}, z_2 + b - a\}$ are separated zeros of u'. This implies that

$$[S\backslash Z(u')] \geqslant m + 1 = d + 1.$$

By a repeated application we finally obtain that

$$[S\backslash Z(u^{(l-q)})] \geqslant d + 1.$$

But this contradicts (4.5), because $u^{(l-q)} \in P_q^0(\Delta)$ and $\dim P_q^0(\Delta) = q(k+1)$.

In the same way we can show that $Z(u) \cap [x_1, x_{m+1}] = \{x_i\}_{i=1}^{m+1}$.

Thus we have shown that u has exactly $m+1$ zeros in $[x_1, x_{m+1}]$ and changes the sign at x_i, $i = 2, \dots, m$. Hence setting $\tilde{u} = \varepsilon u$ for some $\varepsilon\in\{-1, 1\}$ we obtain

$$(-1)^i \tilde{u}(x) \geqslant 0, \quad x\in[x_i, x_{i+1}], \; i = 1, \dots, m.$$

Assume now that $m \leqslant d - 1$ and d is even. Then arguing in the same way as in Case 3.6.1 and applying Claim 4.3 we obtain a $\tilde{u}\in U\backslash\{0\}$ such that (4.11) holds. If d is odd, then by Claim 4.2 U itself is a WT-system on $[a,b]$ and arguing analogously as in the case when d is even a $\tilde{u} \in U \backslash \{0\}$ with the desired properties can be found.

Case 4.4.2: *Assume that u has at least two zero intervals* $J_1 = [e_{\mu_1}, e_{v_1}]$ *and* $J_2 = [e_{\mu_2}, e_{v_2}]$ *in* $[a,b]$ *such that* $e_{v_1} < e_{\mu_2}$ *and u has at most finitely many zeros in* $[e_{v_1}, e_{\mu_2}]$. Let $\{x_i\}_{i=1}^m \cap (e_{v_1}, e_{\mu_2}) = \{y_i\}_{i=1}^r$ such that $y_0 = e_{v_1} < y_1 < \cdots < y_r < e_{\mu_2} = y_{r+1}$. Define $\hat{u} \in V = S_l^{l-q}(\Delta)$ satisfying $\hat{u} = 0$ on $[a, e_{v_1}] \cup [e_{\mu_2}, b]$ and $\hat{u} = u$ on (e_{v_1}, e_{μ_2}). Since V satisfies Property A on $[a,b]$, the subspace $V(\hat{u})$ satisfies Property A on $[a,b]$. Then, considering several cases as in Case 3.6.2 we find a $\tilde{u} \in V(\hat{u}) \backslash \{0\}$ such that (4.11) holds.

Moreover, $\tilde{u}^{(j)}(a) = \tilde{u}^{(j)}(b) = 0, j = 0, \ldots, l-q$. Therefore, $\tilde{u} \in U(u)$.

Case 4.4.3: *Assume that u has a unique zero interval* $J = [e_\mu, e_v]$ *in* $[a,b]$. Then by definition of U, u has an additional zero interval $\tilde{J} = [e_{\mu+k+1}, e_{v+k+1}]$ in the interval $[e_\mu, e_{v+k+1}]$. Since $S_l^{l-q}(\Delta)$ also satisfies Property A on $[a, a+2(b-a)]$, analogously arguing as in Case 4.4.2 we obtain the desired function $\tilde{u}$.

This completes the proof of Claim 4.4. □

Proof of Theorem 8. Let $U = P_l^{l-q}(\Delta)$. If $q = l$, the statement follows from Case 4.1. Otherwise, let $q \in \{1, \ldots, l-1\}$. Since the constant functions are contained in U, it follows that $Z(U) = \emptyset$. Moreover, in view of Claim 4.1, statement (2)(a) of Theorem 5 is satisfied. Finally, statement (2)(b) of Theorem 5 follows from Claim 4.4.

Hence by Theorem 5, $P_l^{l-q}(\Delta)$ satisfies Property A_{per}. □

Remark. For the special case when $q = 1$ and the weight function $w = 1$ it was shown in [3] that every $f \in C_{b-a}$ has a unique L^1-approximation from $U = P_l^{l-1}(\Delta)$.

References

[1] V.F. Babenko, M.E. Gorbenko, On the uniqueness of an element of the best L_1-approximation for functions with values in a Banach space, Ukrainian Math. J. 52 (2000) 29–34.

[2] W. Li, Weak Chebyshev subspaces and A-subspaces of $C[a,b]$, Trans. Amer. Math. Soc. 322 (1990) 583–591.

[3] G. Meinardus, G. Nürnberger, Uniqueness of best L_1-approximations from periodic spline spaces, J. Approx. Theory 58 (1989) 114–120.

[4] A. Pinkus, On L^1-Approximation, Cambridge University Press, Cambridge, 1989.

[5] A. Pinkus, B. Wajnryb, Necessary conditions for uniqueness in L^1-approximation, J. Approx. Theory 53 (1988) 54–66.

[6] L.L. Schumaker, Spline Functions: Basic Theory, Wiley, New York, 1981.

[7] H. Strauss, Best L_1-approximation, J. Approx. Theory 41 (1984) 297–308.